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On a model of laminar–turbulent transition

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In the perturbation theory of a shear flow, a small-lengthscale turbulent perturbation field component developing from the pre-existing turbulence is taken into account along with the usual long-wave (smooth) component. The perturbation turbulence field is assumed to be fully developed, and to satisfy the Kolmogorov-type similarity hypotheses. At the same time the perturbations of the mean velocity field and its gradient due to turbulence are assumed to be small. Under some approximations a closed autonomous set of equations governing the evolution of the turbulent perturbation field can be obtained and qualitatively investigated. The investigation shows in particular that, depending on the initial conditions, the turbulent energy of perturbation can either increase monotonically, or decrease at first and only later start to increase.

Thus, the proposed model of laminar–turbulent transition includes two mechanisms: the usual mechanism of nonlinear self-modulation of long-wave perturbation components, which prevails for small pre-existing turbulence, and the mechanism of the evolution of the small-scale pre-existing turbulence which prevails otherwise. The experimental data are discussed and confirm qualitatively the proposed model.

1. Introduction

The history of turbulence studies usually is reckoned to start with the work of Osborne Reynolds, although two types of fluid motions were clearly distinguished and described by Leonardo da Vinci (c. 1509, see e.g. Popham 1964) and the very term belongs to Lord Kelvin. Moreover, it was Leonardo who described for the first time the coherent structures, so fashionable now. Nevertheless, following tradition we shall start with Reynolds. It was Reynolds who related the turbulence origin to the instability of a laminar fluid motion. He did not, however, identify which instability is dealt with. This left its mark on the further development of the whole theory in a crucial way, as we shall see later. The further development followed the natural path for a perturbation theory in a branch of theoretical physics. The perturbations $\boldsymbol{u}_1(\boldsymbol{x},t)$ and $p_1(\boldsymbol{x},t)$ of velocity and pressure fields were considered to be small and smooth, so the perturbed velocity and pressure fields were represented in the form

$$\left.\begin{aligned} \boldsymbol{u}(\boldsymbol{x},t) &= \boldsymbol{u}_0(\boldsymbol{x})+\boldsymbol{u}_1(\boldsymbol{x},t),\\ p(\boldsymbol{x},t) &= p_0(\boldsymbol{x})+p_1(\boldsymbol{x},t).\end{aligned}\right\} \qquad (1.1)$$

Here $\boldsymbol{u}_0(\boldsymbol{x})$ and $p_0(\boldsymbol{x})$ are velocity and pressure fields corresponding to undisturbed steady motion, and $\boldsymbol{x}(x_1,x_2,x_3)$, t are the coordinate vector and time respectively. Substituting the expressions (1.1) for the disturbed fields into the Navier–Stokes equations and neglecting the term $(\boldsymbol{u}_1\cdot\boldsymbol{\nabla})\,\boldsymbol{u}_1$ because of the smallness and smoothness of the perturbation fields one obtains a well-known linear system of equations of linear stability theory (see e.g. Landau & Lifshitz 1959; Schlichting 1968; Kochin, Kibel & Roze 1948):

$$\left.\begin{aligned} \rho(\partial_t \boldsymbol{u}_1 + (\boldsymbol{u}_0 \cdot \boldsymbol{\nabla})\boldsymbol{u}_1 + (\boldsymbol{u}_1 \cdot \boldsymbol{\nabla})\boldsymbol{u}_0) &= -\boldsymbol{\nabla} p_1 + \boldsymbol{\nabla}\boldsymbol{\tau}_1, \\ \boldsymbol{\nabla} \cdot \boldsymbol{u}_1 &= 0. \end{aligned}\right\} \tag{1.2}$$

Here $\boldsymbol{\nabla}\boldsymbol{\tau}_1 = \rho\nu\nabla^2\boldsymbol{u}_1$ is the divergence of the viscous stress field corresponding to the perturbation of the velocity field, and ν is the kinematic viscosity coefficient. The coefficients of the system (1.2) are time-independent; therefore wave-type solutions appear, depending exponentially on time:

$$\boldsymbol{u}_1 = \mathrm{e}^{-\mathrm{i}\Omega t}\, U(\boldsymbol{x}), \quad p_1 = \mathrm{e}^{-\mathrm{i}\Omega t} P(\boldsymbol{x}). \tag{1.3}$$

Therefore, the spectral problem arises – that of the determination of the complex eigenvalues $\Omega = \Omega_r + \mathrm{i}\Omega_i$, etc. In the simplest formulation, for the perturbations of an arbitrary steady rectilinear shear flow along the x_1 axis in the (x_1, x_2)-plane, the classical Orr–Sommerfeld problem is obtained, namely

$$\left.\begin{aligned} &[V(\zeta)-c]\left(\frac{\mathrm{d}^2 f}{\mathrm{d}\zeta^2}-\alpha^2 f\right)-\left(\frac{\mathrm{d}^2 V}{\mathrm{d}\zeta^2}\right)f = -\left(\frac{\mathrm{i}}{\alpha Re}\right)\left(\frac{\mathrm{d}^4 f}{\mathrm{d}\zeta^4}-2\alpha^2\frac{\mathrm{d}^2 f}{\mathrm{d}\zeta^2}+\alpha^4 f\right), \\ &\text{subject to} \qquad f(1) = f(-1) = f'(1) = f'(-1) = 0. \end{aligned}\right\} \tag{1.4}$$

In these equations α is the dimensionless perturbation wavenumber, Re the Reynolds number, $\zeta = x_2/h$ (it is assumed that the flow is contained between solid walls at $x_2 = -h$ and $x_2 = h$), $c = c_r + \mathrm{i}c_i$ is a dimensionless eigenvalue, f the dimensionless stream function of the perturbation field, and $V(\zeta)$ the lateral distribution of the dimensionless velocity of the undisturbed shear flow.

In the precomputer era the solution to the problem (1.4) appeared to be rather difficult, so its study was not deprived of a certain drama (see Heisenberg 1969).

Analogous problem statements are well known for other flows also: the viscous fluid rotation between cylinders and spheres, convection in a horizontal fluid layer, etc. For these latter flows there are typically a rather small number of vortices in a typical case, so accordingly, in applications to these systems, so-called scenarios of turbulence development, under transition through neutral surfaces in the parameter spaces, were designed, based on the use of low-dimension dynamical systems.

The beginning of linear instability does not, however, mean the transition to turbulence. In this respect the study of laminar–turbulent transition in a boundary layer became specially instructive. Tollmien (1929) and Schlichting (1933) investigated the stability of a rectilinear shear flow with the velocity profile of a boundary layer on a plate. (It was assumed that the longitudinal variation of hydrodynamic fields could be neglected, so this study was assumed to be able to give the answer to the stability problem of the actual boundary-layer flow on a plate.) An instability region in the (α, Re)-plane was found. Since $c_r = c_r(\alpha, Re)$, $c_i = c_i(\alpha, Re)$, $\alpha = \alpha(Re, c_r)$, we can determine (prescribing the dimensionless oscillation frequency c_r and Reynolds number Re) the wavelength of the oscillations arising at a prescribed frequency: the instability under periodical forcing should appear in the form of waves.

More than ten years later experimenters were still unable to observe these waves – the Tollmien–Schlichting waves – until H. Dryden understood the reason: the free-stream flow was too strongly turbulent even at the entrance to the working section of the wind tunnel. Under Dryden's guidance a specialized 'low-turbulence' wind tunnel was designed and his co-workers at the US National Bureau of Standards, Schubauer & Skramstad (1947) were able to observe the Tollmien–Schlichting wave. A recent photograph of Tollmien–Schlichting waves taken by Werlé can be found in the remarkable *Album of Fluid Motion* by Van Dyke (1981).

In low-turbulence wind tunnels the secondary two- and three-dimensional instabilities of the Tollmien–Schlichting waves themselves can also be observed; these are essentially nonlinear processes, leading ultimately to turbulence.

Let us try to formulate, however, a view as to what precisely Dryden and Schubauer & Skramstad achieved in this work. Schlichting (1968) writes about it as follows: 'The experimental results reported in this Chapter show such complete agreement with the theory of stability of laminar flows that the latter may now be considered as a verified component of fluid mechanics. The hypothesis that the process of transition from laminar to turbulent flow is the consequence of an instability in the laminar flow, enunciated by O. Reynolds, is hereby completely vindicated. It certainly represents a *possible* and *observable* mechanism of transition. *The question as to whether it paints a complete picture of the process and whether it constitutes the only mechanism encountered in nature is still at present an open one*' (the italics are mine – G. B.). The last sentence is very instructive; it is missing in the first editions of Schlichting's book. In fact, Dryden, Schubauer and Skramstad showed, by their experiments in a low-turbulence wind tunnel, seemingly only that the Navier–Stokes equations are correct. At that time this was important, because the inability to observe the Tollmien–Schlichting waves provided doubts even on that point. However, from today's viewpoint this is no more than a small piece from the history of this branch of hydromechanics.

Indeed, for the observation of the Tollmien–Schlichting waves, a very low turbulence level u'/U at the entrance to the working section was required (u' is the root-mean-square velocity fluctuation, and U the mean flow velocity). For a turbulence level even one order of magnitude higher than in low-turbulence wind tunnels, although the disturbances remain very small, the perturbation theory is no longer valid. The results of Dryden, Schubauer and Skramstad suggest in fact that the classical approach mentioned above is fundamentally insufficient.

It is useful to recall here the classical experiments of S. Kline's Stanford group (see Offen & Kline 1975). These experiments confirmed clearly that turbulent flow contains a vortex cascade, whose interactions produce turbulent 'bursts' – spots of quite fully developed turbulence. Their analysis of turbulent shear flows showed that the turbulence generation is entirely determined by these bursts. Therefore, in a perturbed flow the existence of small-lengthscale vortices in the flow from the very beginning should be taken into account, as well as the turbulence generated by their interaction. This is the purpose of the present work. We assume that, in spite of its small lengthscale and low energy, the perturbation turbulence is fully developed and the Kolmogorov (1942) similarity hypotheses can be applied. A closed set of equations for the turbulence properties can be obtained and investigated qualitatively. Thus, the existence of two mechanisms of laminar–turbulent transition is clearly demonstrated: the usual evolution of long-wave perturbations with ultimate transition to turbulence, and the evolution of small-scale low-energy turbulence present in the flow from the very beginning.

2. Basic relations of the model

As stated in §1, the properties of the perturbed flow are represented as a sum of those for the basic undisturbed steady flow, small long-wave perturbations of a standard type, and small-lengthscale and low-energy turbulent perturbations. The turbulent perturbation is represented as a cascade of small-lengthscale vortices whose energy is small in comparison with the energy of the basic flow. Since the lengthscale of the vortices is small in comparison with that of the basic flow, the field of the turbulent vortices can be considered as isotropic. The velocity and pressure fields of the disturbed flow are represented, consequently, not in the form (1.1) but in the form

$$\boldsymbol{u} = \boldsymbol{u}_0(\boldsymbol{x})+\boldsymbol{u}_1(\boldsymbol{x},t)+\boldsymbol{u}'(\boldsymbol{x},t), \tag{2.1}$$

$$p = p_0(\boldsymbol{x})+p_1(\boldsymbol{x},t)+p'(\boldsymbol{x},t). \tag{2.2}$$

Here $\boldsymbol{u}'(\boldsymbol{x},t)$, $p'(\boldsymbol{x},t)$ are the velocity and pressure fields of small-lengthscale turbulence present, by assumption, in the flow from the very beginning and developing in time and space.

Substituting (2.1) and (2.2) into the Navier–Stokes and continuity equations we obtain, after averaging over the ensemble of realizations of the vortex field (the bar denoting this probabilistic averaging), as is commonly done in turbulence studies,

$$\rho(\partial_t \boldsymbol{u}_1+(\boldsymbol{u}_0\cdot\boldsymbol{\nabla})\boldsymbol{u}_1+(\boldsymbol{u}_1\cdot\boldsymbol{\nabla})\boldsymbol{u}_0) = -\boldsymbol{\nabla}p_1+\boldsymbol{\nabla}(\boldsymbol{\tau}_1-\rho\overline{\boldsymbol{u}'\boldsymbol{u}'}-\rho\boldsymbol{u}_1\boldsymbol{u}_1), \tag{2.3a}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{u}_1 = 0. \tag{2.3b}$$

Thus, under the divergence sign on the right-hand side of (2.3a) there is, besides the viscous stress $\boldsymbol{\tau}_1$ from the long wave disturbance, the viscous stress tensor $-\rho\overline{\boldsymbol{u}'\boldsymbol{u}'}$ stipulated by turbulent perturbations, with components $-\rho\overline{u'_i u'_j}$, and the stress tensor $-\rho\boldsymbol{u}_1\boldsymbol{u}_1$, with the components $-\rho u_{1i}u_{1j}$, stipulated by the contribution of the nonlinear self-interaction of long-wave perturbations. As is seen, the set (2.3) is not closed.

For closing this set we note an important point. According to our basic assumption the turbulence in the flow is of small lengthscale, i.e. its lengthscale is small in comparison with the characteristic lengthscale of the flow. It is natural therefore to take another basic assumption, of Kolmogorov-type self-similarity, concerning the turbulent perturbation: the structure of the vortex cascade field at all points of the flow is the same: only the vortex size and the energy may vary. This assumption means that all dimensionless properties of the vortex cascade field are the same, and consequently all statistical properties of the turbulence field are completely determined by the fluid density and two kinematic properties of different dimensions. As our choice of these kinematic properties, we take the turbulent energy per unit mass $b = \frac{1}{2}(\overline{u_1'^2}+\overline{u_2'^2}+\overline{u_3'^2})$ and the dissipation rate of turbulent energy into heat per unit mass ϵ. A hypothesis of this type but in slightly different form was first proposed by Kolmogorov (1942); now it forms the basis of modern semi-empirical turbulence theories. In particular this hypothesis is essentially the basis of the well-known b–ϵ model of semi-empirical turbulence theory (in the literature the notation k is more often used instead of b, as well as the term k–ϵ model; we prefer the original Kolmogorov notation). Let us note that, under the conditions of a laminar flow weakly disturbed by fully developed, although small-lengthscale and low-energy, turbulence† the Kolmogorov self-similarity hypothesis seems to be the most

† The analogy with adult Lilliputians splendidly developed by Jonathan Swift (1726) seems to be appropriate here: they are small but they have everything that ordinary adult people have.

appropriate one. For closing the system (2.3) the equations of turbulent energy and turbulent energy dissipation rate should be used. These equations are obtained, in the usual way, from the Navier–Stokes equations by multiplying by velocities, velocity gradients, etc., followed by probabilistic averaging and certain algebraic transformations (see e.g. Reynolds 1976).

On account of our basic assumption concerning the universal structure of the turbulence field of the disturbance, all the kinematic statistical properties of the vortex field are determined by two of them: we have selected b and ϵ. The dimensionless properties of the vortex field are universal. At the same time, owing to the small turbulence lengthscale, the isotropy of the turbulence and the smallness of the long-wave disturbance, the contributions of all perturbations (long-wave and turbulent ones) to the mean velocity field and its gradient can be neglected. Therefore in (2.3) the mean velocity field and its gradient can be taken from the undisturbed flow: steady rectilinear shear flow along the x_1-axis with the velocity varying in the x_2-direction, $u_0 = u_0(x_2)$.

The equations of turbulent energy and dissipation rate balance take the forms, when we neglect further the contributions of molecular viscosity (Reynolds 1976),

$$\partial_t b + u_0 \partial_1 b = \alpha_1 \nabla \cdot \left(\frac{b^2}{\epsilon} \nabla b\right) + \alpha \frac{b^2}{\epsilon} (\partial_2 u_0)^2 - \epsilon, \tag{2.4}$$

$$\partial_t \epsilon + u_0 \partial_1 \epsilon = \beta \nabla \cdot \left(\frac{b^2}{\epsilon} \nabla \epsilon\right) + \delta b (\partial_2 u_0)^2 - \gamma \frac{\epsilon^2}{b}. \tag{2.5}$$

Here α, α_1, β, γ, δ are universal constants according to our basic assumption about the self-similarity of turbulent perturbations.

Comparison with experimental data for some special flows has given the following values for some of these constants (Reynolds 1976):

$$\alpha = 0.07; \quad \gamma = 2; \quad \delta = 0.08. \tag{2.6}$$

Equations (2.4) and (2.5) are, naturally, simplified relations of the b–ϵ semi-empirical turbulence model, often applied to the calculation of turbulent flows. Note that in the case under consideration, when the turbulent perturbation is a low-energy, small-lengthscale and high-frequency field, the universality of the constants is a more rigorous implication of the adopted hypothesis. In contrast to the fully developed turbulent flows, where the turbulence lengthscales are comparable with the global lengthscale of the flow and the contribution of turbulence to the mean velocity field is a significant one, the contribution of turbulence to the mean velocity and its gradient field is negligible here.

Accordingly, the mean velocity gradient field can be considered as given beforehand: it is the velocity gradient field of the undisturbed flow. This very property makes the system (2.4), (2.5) a closed set of equations for two unknowns, b and ϵ.

For final closure of the set (2.3) it is necessary to complement the autonomous set of equations (2.4), (2.5) by a relation for Reynolds stress which, under the assumption adopted of isotropy of turbulent vortices, can be written in the form

$$\rho \overline{\boldsymbol{u}'\boldsymbol{u}'} = -\tfrac{2}{3}\rho b \boldsymbol{I} + \alpha \rho \frac{b^2}{\epsilon} \boldsymbol{D}, \tag{2.7}$$

where $\boldsymbol{D}$ is the strain rate tensor corresponding to undisturbed flow. For the shear flow under consideration $D_{12} = \partial_2 u_0$, and all remaining components D_{ij} are equal to zero. This fact was used during the derivation of the system (2.4), (2.5).

3. Simplification and qualitative analysis of the system (2.4), (2.5)

Let us reduce the system (2.4), (2.5) to a dimensionless form, writing

$$b = u_0^2 B(\zeta, \xi), \quad \epsilon = \frac{u_0^3}{\lambda} E(\zeta, \xi), \quad u_0 = UV(\zeta), \quad \zeta = \frac{x_2}{L}, \quad \xi = \frac{x_1}{L}, \quad \boldsymbol{\xi} = \frac{\boldsymbol{x}}{L}, \quad \tau = \frac{Ut}{L}. \tag{3.1}$$

Here L is the characteristic flow lengthscale, e.g. its width h, u_0 is an initial velocity scale of the turbulent perturbation, and λ is an initial lengthscale of the turbulent perturbation. Let us consider a steady state in the mean turbulent flow in which the perturbations are developing in the longitudinal x_1-direction. The equations (2.4) and (2.5) take the form

$$V(\zeta)\frac{\mathrm{d}B}{\mathrm{d}\xi} = \alpha_1 \frac{u_0 \lambda}{UL} \nabla_\xi \cdot \left(\frac{B^2}{E} \nabla_\xi B\right) + \alpha K \left(\frac{\mathrm{d}V}{\mathrm{d}\zeta}\right)^2 \frac{B^2}{E} - \frac{E}{K}, \tag{3.2}$$

$$V(\zeta)\frac{\mathrm{d}E}{\mathrm{d}\xi} = \beta \frac{u_0 \lambda}{UL} \nabla_\xi \cdot \left(\frac{B^2}{E} \nabla_\xi E\right) + \delta K \left(\frac{\mathrm{d}V}{\mathrm{d}\zeta}\right)^2 B - \gamma \frac{E^2}{KB}, \tag{3.3}$$

where $$K = \lambda U / L u_0.$$

It is clear that the parameter $u_0 \lambda / UL$ is a product of two small quantities; therefore we can neglect† the divergence terms, and the basic system of equations for the dimensionless properties of a turbulent flow, (3.2) and (3.3), takes an unexpectedly simple form,

$$V(\zeta)\frac{\mathrm{d}B}{\mathrm{d}\xi} = \alpha \left(K \left(\frac{\mathrm{d}V}{\mathrm{d}\zeta}\right)^2\right) \frac{B^2}{E} - \frac{E}{K}, \tag{3.4}$$

$$V(\zeta)\frac{\mathrm{d}E}{\mathrm{d}\xi} = \delta \left(K \left(\frac{\mathrm{d}V}{\mathrm{d}\zeta}\right)^2\right) B - \frac{\gamma E^2}{KB}. \tag{3.5}$$

In (3.4) and (3.5) $K = (\lambda/L)/(u_0/U)$ is the basic constant parameter for the problem under consideration which, in principle, can admit arbitrary positive values – it is the ratio of two small parameters. The system (3.4), (3.5) allows one to carry out a complete qualitative investigation in the (B, E)-plane, and can be reduced to a single quadrature.

Let us consider first a degenerate case, $\mathrm{d}V/\mathrm{d}\zeta = 0$ (a shear-free flow). In this case the system (3.4), (3.5) is reduced to a simple equation

$$\frac{\mathrm{d}B}{\mathrm{d}E} = \frac{B}{\gamma E}, \tag{3.6}$$

whence $B = \text{const.} \times E^{1/\gamma}$. Substituting this relation into (3.4) we obtain, by integrating for large ξ,

$$E \sim \xi^{-\gamma/(\gamma-1)} \to 0, \quad B \sim \xi^{-1/(\gamma-1)} \to 0, \tag{3.7}$$

i.e. a natural result – that for the decay of turbulent disturbances in a shear-free flow. Furthermore, the current lengthscale of the turbulent perturbation is determined by the relation

$$l = \lambda B^{\frac{3}{2}}/E \times \text{const.}, \tag{3.8}$$

† Taking into account these terms gives no difficulties of principle. However, the simplified system is instructive and allows the most effective qualitative investigation, so it should be considered in the first place.

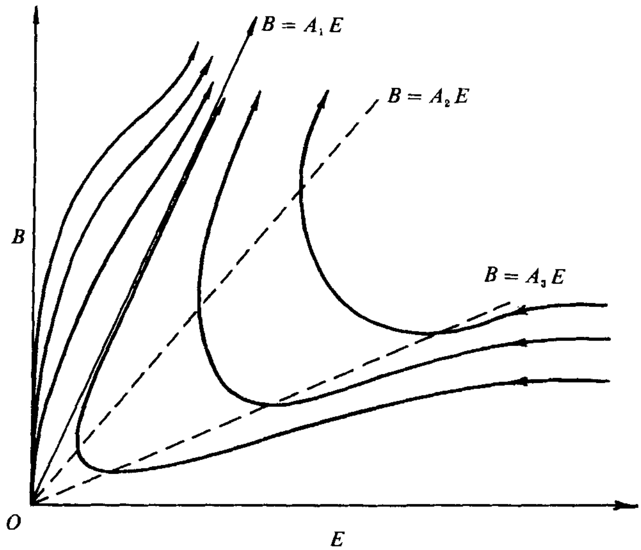

FIGURE 1. The qualitative investigation of integral curves in the phase plane of dimensionless turbulent energy – dimensionless dissipation rate.

whence, as $\xi \to \infty$, it follows according to (3.7) that

$$l \sim \lambda \xi^{(\gamma - \frac{3}{2})/(\gamma - 1)} \to \infty. \tag{3.9}$$

Therefore at large distances from the entrance to the working section of the wind tunnel the turbulence lengthscale becomes comparable with the flow lengthscale and the proposed model becomes invalid.

In the non-degenerate case $\mathrm{d}V/\mathrm{d}\zeta \neq 0$, we denote

$$y = K \left|\frac{\mathrm{d}V}{\mathrm{d}\zeta}\right| \frac{B}{E}. \tag{3.10}$$

Dividing (3.5) by (3.6) we reduce the system (3.5), (3.6) to a single quadrature,

$$E \frac{\mathrm{d}y}{\mathrm{d}E} = y \left[\frac{\alpha y^2 - 1}{\delta y^2 - \gamma} - 1\right]. \tag{3.11}$$

Taking into account the numerical values of the parameters (2.8) we obtain the phase picture in the (B, E)-plane represented in figure 1. In fact, dividing (3.5) by (3.6) we obtain

$$\frac{\mathrm{d}B}{\mathrm{d}E} = \frac{B}{E} \frac{\alpha(K^2(\mathrm{d}V/\mathrm{d}\zeta)^2) B^2 - E^2}{\delta(K^2(\mathrm{d}V/\mathrm{d}\zeta)^2) B^2 - \gamma E^2}, \tag{3.12}$$

so an invariant straight line exists:

$$B = A_1 E, \quad A_1 = \left(\frac{\gamma - 1}{\delta - \alpha}\right)^{\frac{1}{2}} \left(K\left(\frac{\mathrm{d}V}{\mathrm{d}\zeta}\right)\right)^{-1}. \tag{3.13}$$

Furthermore, on the straight line

$$B = A_2 E, \quad A_2 = \left(\frac{\gamma}{\delta}\right)^{\frac{1}{2}} \left(K\left(\frac{\mathrm{d}V}{\mathrm{d}\zeta}\right)\right)^{-1} < A_1, \tag{3.14}$$

the derivative $\mathrm{d}B/\mathrm{d}E$ becomes infinite and changes its sign: this straight line is an infinity isocline. The integral curves under the straight line $B = A_2 E$ behave as $E \to \infty$ according to $B = \text{const.} \times E^{1/\gamma}$. Finally, on the straight line

$$B = A_3 E, \quad A_3 = \alpha^{-\frac{1}{2}}\left(K\left(\frac{dV}{d\zeta}\right)\right)^{-1} < A_2, \tag{3.15}$$

which is the zero isocline, these curves reach their minima.

Figure 1 shows that, depending on the initial values of B and E, i.e. ultimately on the initial intensity and initial lengthscale of turbulent perturbations in a shear flow, the integral curves can be divided into two classes. On curves of the first class the quantity B, i.e. the turbulent energy, increases monotonically. On curves of the second class, B decreases at first and later, after reaching a certain minimum, starts to increase.

4. Discussion and conclusion

The simple model presented above confirms the prediction of Schlichting. The basic conclusion is that during laminar–turbulent transition in a shear flow there occurs, generally speaking, a competition between two mechanisms: (1) nonlinear instability of the usual type, and (2) the development of pre-existing turbulence. The first mechanism can be observed in its pure form when the pre-existing turbulence is sufficiently small. Mathematically it is expressed by the smallness of the contribution of the Reynolds stress $-\rho\overline{\boldsymbol{u}'\boldsymbol{u}'}$ to the divergence term of equation (2.3a)

$$\nabla(\tau_1 - \rho \boldsymbol{u}_1 \boldsymbol{u}_1 - \rho\overline{\boldsymbol{u}'\boldsymbol{u}'}) \tag{4.1}$$

in comparison with the term $\rho \boldsymbol{u}_1 \boldsymbol{u}_1$.

This particular mechanism is realized in low-turbulence wind tunnels: the proposed model can estimate quantitatively the limits of applicability of the classical approach. On the other hand, in the case when the evolution of the pre-existing turbulence plays the basic role, two types of development are possible. In the first case the turbulent energy increases monotonically, in the second one it decreases at first as in the experiments of Batchelor & Townsend (1949) on the decay of grid turbulence, and only later starts to increase. In some sense the proposed model is an alternative to the classical (Batchelor 1953) rapid distortion model.

The model presented above is in qualitative accord with available experimental results: the occurrence of two different mechanisms of laminar–turbulent transition for different intensities of pre-existing turbulence in a boundary layer was noted in the paper by Kolyada & Paveliev (1986). The first mechanism, observed for small intensities of the pre-existing turbulence, corresponds to the growth of disturbances of a certain frequency (i.e. nonlinear self-excitation of long-wave disturbances). The second mechanism, observed for large pre-existing turbulence intensities, corresponds to the development of a continuous spectrum without distinguished frequencies. Furthermore, in a recent paper by Rohr *et al.* (1988) the development of small turbulent disturbances in a shear flow with a constant shear rate was investigated. In that paper, instructive examples of the turbulent energy variation corresponding to both types indicated above can be found: the first as in figure 2, and second as in figure 3.

The model presented above has obvious restrictions. First of all, the turbulence lengthscale increases with distance from the entrance section. When this lengthscale becomes comparable with the external flow lengthscale the model becomes, strictly speaking, invalid. Moreover, the question remains as to whether the turbulence in

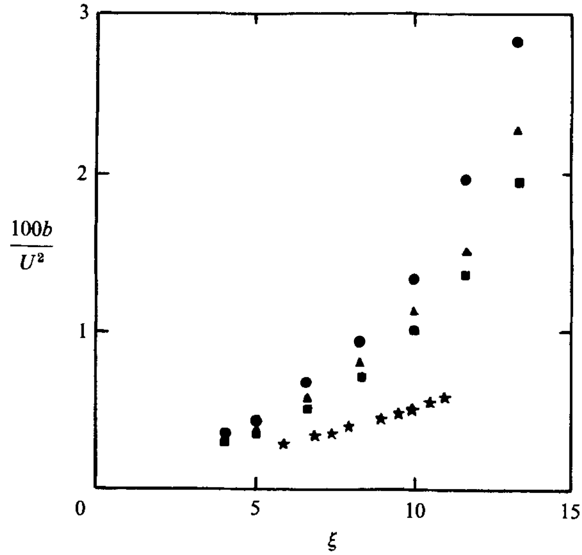

FIGURE 2. Experimental results (Rohr *et al.* 1988, figure 8*a*) showing the first type of turbulence evolution: the turbulent energy grows monotonically with downstream distance.

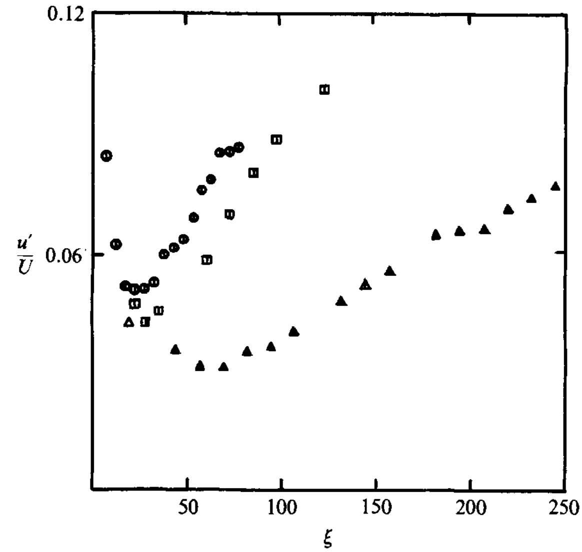

FIGURE 3. Experimental results from a water tunnel (Rohr *et al.* 1988, figure 6) showing the second type of turbulence evolution: the turbulent energy at first decreases and later starts to increase with downstream distance.

the perturbed flow can be always considered as fully developed and whether the Kolmogorov-type similarity hypotheses remain applicable in their simplest form. We note, however, that there exist more sophisticated models which can allow us to refine the quantitative calculations of turbulent perturbations.

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